ON THE SOLVABILITY OF GENERAL PROBLEMS FOR AN ELASTIC CLOSED CYLINDRICAL SHELL IN A NONLINEAR FORMULATION

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I. I. VOROVICH and G. A. KOSUSHKIN
(Rostov-on-Don, Moscow)
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Mathematical aspects of nonlinear shallow shell theory are examined in [1-3] (*). Shallow shell relationships cannot always be utilized in a study of the problem of the equilibrium modes of a closed cylindrical shell within the scope of the nonlinear theory of "mean" flexure [3,4]. A theoretical investigation of the nonlinear equations for a cylindrical shell, obtained without utilizing the shallowness hypothesis, is presented herein. The existence of a generalized solution of the nonlinear problem for an arbitrary loading, and arbitrary support conditions is proved.

1. Fundamental relationships. Formulation of the problem. Let us examine the following modification of the relationships of the nonlinear theory for a cylindrical shell which are easily obtained from the relationships for the mean flexure [4]:

$$\varepsilon_1 = u_x + {}^1/_2\, w_x^2, \qquad \varepsilon_2 = v_y + kw + {}^1/_2\,(w_y - kv)^2 \tag{1.1}$$

$$2\varepsilon_{12} = u_y + v_x + w_x(w_y - kv)$$

$$\varkappa_1 = -w_{xx}, \qquad \varkappa_2 = -w_{yy} + kv_y, \qquad \varkappa_{12} = -w_{xy} + kv_x$$

$$T_1 = 2B_1(\varepsilon_1 + \nu\varepsilon_2), \qquad T_2 = 2B_1(\nu\varepsilon_1 + \varepsilon_2), \qquad T_{12} = B_1(1-\nu)2\varepsilon_{12}$$

$$M_1 = 2B_2(\varkappa_1 + \nu\varkappa_2), \qquad M_2 = 2B_2(\nu\varkappa_1 + \varkappa_2), \qquad M_{12} = 2B_2(1-\nu)\varkappa_{12}$$

$$B_1 = \frac{Eh}{2(1-\nu^2)}, \quad B_2 = \frac{Eh^3}{24(1-\nu^2)}, \quad B_{12} = \frac{B_1}{B_2}, \quad B_{21} = \frac{B_2}{B_1}, \quad 0 < \nu < 0.5$$

The following notation is utilized in (1.1): u, v, w are displacements of points of the middle surface; the subscript x, y on the u, v, w denotes differentiation with respect to x, y, respectively; $\varepsilon_1, \varepsilon_2, \varepsilon_{12}$ are the tensile and shear strains, and $\varkappa_1, \varkappa_2, \varkappa_{12}$ changes in the curvature of the shell middle surface; T_1, T_2, T_{12} the stress resultants in the plane of the shell; N_1, N_2 transverse forces; M_1, M_2, M_{12} the bending moments and torque; E, ν the elastic constants of the material; h the thickness, and k the shell curvature. The x-axis is directed along the cylinder generator, the y-axis along the tangent to the directrix, and the z-axis along the normal to the middle surface.

The shell planform occupies a domain G with boundary Γ.

$$G = \{(x,y) : |x| < l_1, |y| < l_2\}, \qquad \Gamma = \Gamma_1 \cup \Gamma_2, \qquad \Gamma_1 = \Gamma_1^1 \cup \Gamma_1^2$$

$$\Gamma_1^1 = \{(x,y) : x = -l_1, |y| \leqslant l_2\}, \qquad \Gamma_1^2 = \{(x,y) : x = l_1, |y| \leqslant l_2\}$$

$$\Gamma_2 = \Gamma \setminus \Gamma_1 \qquad (kl_2 = \pi)$$

*) See also the Doctoral Dissertation of I. I. Vorovich.

Here l_1 is half the shell length. Evidently Γ_1^1, Γ_1^2 are the left and right endfaces of the shell, respectively.

The state of stress in a normal section of the shell is characterized by four quantities: T_n the stress resultant normal to the contour in the plane of the shell; S_n the tangential; N_n the transverse stress resultant; and M_n the bending moment. The equilibrium differential equations in the stress resultants are the following:

$$\frac{\partial T_1}{\partial x} + \frac{\partial T_{12}}{\partial y} - (X - w_x Z_1) = 0$$
$$\frac{\partial T_{12}}{\partial x} + \frac{\partial T_2}{\partial y} + kT_2 (w_y - kv) + kT_{12} w_x + k\left(\frac{\partial M_2}{\partial y} + \frac{\partial M_{12}}{\partial x}\right) - [Y - (w_y - kv) Z_1] = 0 \quad (1.2)$$
$$\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 M_{12}}{\partial x \partial y} + \frac{\partial^2 M_2}{\partial y^2} + \frac{\partial}{\partial x}(T_1 w_x) + \frac{\partial}{\partial y}(T_{12} w_x) +$$
$$+ \frac{\partial}{\partial x}[T_{12}(w_y - kv)] + \frac{\partial}{\partial y}[T_2 (w_y - kv)] - kT_2 - (Z + Z_1) = 0$$

Here X, Y, Z are surface loading components in the x, y, z coordinate system whose direction is independent of the strain· Z_1 the normal following loading (hydrostatic pressure). Taking account of the X_1, Y_1 components within the scope of mean bending theory is inconsistent.

Let the geometric and static boundary conditions be given on the sets γ_i^- and γ_i^+ $(i = 1, 2, 3, 4)$, respectively. Evidently $\gamma_i^+ \subset \Gamma_1$, and $\bar{\gamma}_i \supseteq \Gamma_1 \setminus \gamma_i^+$ $(i = 1,2,3,4)$, since the support may be at internal points of the domain G. The following boundary conditions are considered on Γ_1:

$$u|_{\gamma_1^-} = 0, \quad v|_{\gamma_2^-} = 0, \quad w_x|_{\gamma_4^-} = 0, \quad w|_{\gamma_3^-} = 0 \quad (1.3)$$
$$\{T_n - (T^\circ - k_1 u)\}|_{\gamma_1^+} = 0, \qquad \{S_n - (S^\circ - k_2 v)\}|_{\gamma_2^+} = 0 \quad (1.4)$$
$$\{N_n - (N^\circ - k_3 w)\}|_{\gamma_3^+} = 0, \qquad \{M_n - (M^\circ - k_4 w_x)\}|_{\gamma_4^+} = 0$$

Here $k_1(y)$, $k_2(y)$, $k_3(y)$, $k_4(y)$ are characteristic of the elastic support; T°, S°, N°, M° the external loading on the shell endfaces: the tensile, shear, transverse stress resultants, and the bending moment, respectively. The boundary conditions on Γ_2 are:

$$u, v, w, w_y, T_n, S_n, N_n, M_n|_{y=-l_2}^{y=l_2} = 0 \quad (1.5)$$

Henceforth, we shall study the boundary value problem (1.2) - (1.5). After having substituted (1.1) into (1.2), we obtain the following equilibrium equations in terms of displacements:

$$u_{xx} + \frac{1-\nu}{2} u_{yy} + \frac{1+\nu}{2} v_{xy} = -\frac{\partial}{\partial x}\left[\nu kw + {}^1/_2 w_x^2 + \frac{\nu}{2}(w_y - kv)^2\right] -$$
$$- \frac{(1-\nu)}{2}\frac{\partial}{\partial y}\left[w_x (w_y - kv)\right] + \frac{[X - w_x Z_1]}{2B_1} \equiv f_1$$
$$\frac{1+\nu}{2} u_{xy} + \frac{1-\nu}{2}(1 + 4a^2) v_{xx} + (1 + a^2) v_{yy} - \frac{(2-\nu)}{k} a^2 w_{xxy} -$$
$$- \frac{a^2}{k} w_{yyy} = -\frac{(1-\nu)}{2}\frac{\partial}{\partial x}[w_x (w_y - kv)] - \frac{\partial}{\partial y}\left[\frac{\nu}{2} w_x^2 + kw +\right.$$

$$+\frac{1}{2}(w_y-kv)^2\Big]-k(w_y-kv)\Big[\nu u_x+\frac{\nu}{2}w_x^2+v_y+kw+$$
$$+\frac{1}{2}(w_y-kv)^2\Big]-\frac{(1-\nu)}{2}kw_x[u_y+v_x+w_x(w_y-kv)]+$$
$$+\frac{[Y-(w_y-kv)Z_1]}{2B_1}\equiv f_2$$
$$B_{21}\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}-(2-\nu)kv_{xxy}-k\frac{\partial^3 v}{\partial y^3}\right]=$$
$$=\frac{\partial}{\partial x}\left\{w_x\left[u_x+\frac{1}{2}w_x^2+\nu v_y+\nu kw+\frac{\nu}{2}(w_y-kv)^2\right]\right\}-$$
$$-k\left[\nu u_x+\frac{\nu}{2}w_x^2+v_y+kw+\frac{1}{2}(w_y-kv)^2\right]+\frac{\partial}{\partial y}\{(w_y-kv)\times$$
$$\times[\nu u_x+\frac{\nu}{2}w^2{}_x+v_y+kw+\frac{1}{2}(w_y-kv)^2]\}+\frac{1}{2}(1-\nu)\frac{\partial}{\partial x}\{(w_y-kv)[u_y+$$
$$+v_x+w_x(w_y-kv)]\}+\frac{(1-\nu)}{2}\frac{\partial}{\partial y}\{w_x[u_y+v_x+$$
$$+w_x(w_y-kv)]\}-\frac{(Z+Z_1)}{2B_1}\equiv f_3\quad(a^2=k^2B_{21})\tag{1.6}$$

The equations on the boundary are

$$u|_{\gamma_1^-}=0,\qquad u_x+\nu v_y=-\nu kw-{}^1/_2[w_x^2+\nu(w_y-kv^2)]+$$
$$+(2B_1)^{-1}(T^\circ-k_1u)\equiv\Phi_1\quad \text{на } \gamma_1^+$$
$$v|_{\gamma_2^-}=0,\qquad u_y+(1+4a^2)v_x-\frac{4}{k}a^2w_{xy}=$$
$$=(S^\circ-k_2v)[B_1(1-\nu)]^{-1}-w_x(w_y-kv)\equiv\Phi_2\quad \text{на } \gamma_2^+$$
$$w|_{\gamma_3^-}=0,\qquad \frac{\partial}{\partial x}[w_{xx}+(2-\nu)(w_{yy}-kv_y)]=(2B_2)^{-1}(N^\circ-k_3w)+$$
$$+B_{12}\{w_x[u_x+{}^1/_2w_x^2+\nu v_y+\nu kw+{}^1/_2\nu(w_y-kv)^2]\}+$$
$$+B_{12}{}^1/_2(1-\nu)\{(w_y-kv)[u_y+v_x+w_x(w_y-kv)]\}\equiv\Phi_3\quad \text{на } \gamma_3^+$$
$$w_x|_{\gamma_4^-}=0,\quad w_{xx}+\nu(w_{yy}-kv_y)=-(2B_2)^{-1}(M^\circ-k_4w_x)\equiv\Phi_4\quad \text{на } \gamma_4^+$$
$$u,\ v,\ w,\ u_y,\ v_y,\ w_y,\ w_{yy},\ w_{yyy}\Big|_{y=-l_2}^{y=l_2}=0\tag{1.7}$$

Before investigating the problem (1.6) - (1.7), let us introduce some auxiliary concepts, and let us prove some propositions.

2. Auxiliary propositions. Let $\omega=(u,v,w)$ denote a vector function with components u, v, w. Let us introduce the bilinear form

$$A(\omega_1,\omega_2)=2B_1\int_G\{(u_{1,x}+\nu v_{1,y}+\nu kw_1)u_{2,x}+(\nu u_{1,x}+v_{1,y}+kw_1)(v_{2,y}+kw_2)+$$
$$+{}^1/_2(1-\nu)(u_{1,y}+v_{1,x})(u_{2,y}+v_{2,x})+\alpha[w_{1,x}w_{2,x}+(w_{1,y}-kv_1)(w_{2,y}-kv_2)]+$$
$$+B_{21}[(w_{1,xx}+\nu w_{1,yy}-\nu kv_{1,y})w_{2,xx}+(\nu w_{1,xx}+w_{1,yy}-kv_{1,y})(w_{2,yy}-kv_{2,y})+$$
$$+2(1-\nu)(w_{1,xy}-kv_{1,x})(w_{2,xy}-kv_{2,x})]\}\,dG\quad(\alpha>0)\tag{2.1}$$

It is easy to see that if $A(\omega,\omega)=0$, then

$$\varepsilon_1=\varepsilon_2=\varepsilon_{12}=\varkappa_1=\varkappa_2=\varkappa_{12}=0,\qquad w_x=w_y-kv=0$$

This means that ω is the displacement of the shell as a rigid body.

Let us introduce some functional spaces. A class of functions given in a strip $|x| < l_1$, which is periodic in y with period $2l_2$, can, depending on the metric introduced in it, result in different functional spaces. Hence, because of periodicity, the norm may be computed by means of the periodicity rectangle in y, the domain G. In contrast to the customary spaces $C(G)$, $L_p(G)$, $W_p^{(r)}(G)$, let us supply the space with the degree symbol in the case of periodicity in y The most important properties of the above-mentioned classes of functions carry over completely to the case of partial or total periodicity. In particular, the space $W_p^{(r)}(G^\circ)$ is completely analogous to the Sobolev space, and the same kinds of imbedding theorems [*] are valid for it as for the classes of $W_p^{(r)}(G)$. The norm $W_p^{(r)}(G^\circ)$ is denoted by $\| \ \|_{r,p,G}$, and the norm in $G(G^\circ)$ by $|\cdot|$ Several other notations are elucidated below.

$$\| \ \|_{r,p,G} = \| \ \|_{r,p}, \quad \| \ \|_{0,p,\gamma_i^+} = \|f\|_{p,i}, \quad |f|_{C(\gamma_i^+)} = |f|_i, \quad \|f\|_{0,p,G}^p = \int_G |f|^p dG$$

In this notation

$$\|f\|_{1,2}^2 = \| \ \|_{0,2}^2 + \| \ _x\|_{0,2}^2 + \| \ _y\|_{0,2}^2,$$
$$\|f\|_{2,2}^2 = \|f\|_{0,2}^2 + \| \ _{xx}\|_{0,2}^2 + 2\| \ _{xy}\|_{0,2}^2 + \| \ _{yy}\|_{0,2}^2$$

Let E, the closure of all smooth vector functions ω in the strip $|x| < l_1$ which are periodic with period $2l_2$ in y and satisfy the geometric boundary conditions (1.3), ω denote the closure in the norm of the product of the spaces

$$W_2^{(1)}(G^\circ) \times W_2^{(1)}(G^\circ) \times W_2^{(2)}(G^\circ)$$

As usual, the norm is defined as the direct product

$$\|\omega\|_E^2 = \|u\|_{1,2}^2 + \|v\|_{1,2}^2 + \|w\|_{2,2}^2$$

Let $M \subset E$ be a linear set of all elements $w \in E$ for which $A(\omega, \omega) = 0$. Combining those elements ω' and ω'' equal to M in absolute value, i.e. $\omega' - \omega'' \in M$, into a single class ω, we arrive at the space of classes $E^* = E/M$, a factor space of the space E over the subspace M. In other words, the displacement ω is defined to the accuracy of displacement of the shell as a rigid body under the constraint (1.3). By definition, the factor-space is

$$\|\omega\|_{E^*} = \inf \|\omega'\|_E, \qquad \omega' \in \omega$$

It is easy to see that there exists a unique "normal" representative ω^* of the class ω such that

$$\|\omega\|_{E^*} = \|\omega^*\|_E$$

The space E is a Hilbert space, hence E^* is also a Hilbert space, and therefore

$$(\omega_1, \omega_2)_{E^*} = \inf (\omega_1', \omega_2'')_E, \ \omega_1' \in \omega_1, \qquad \omega_2'' \in \omega_2$$

Lemma 2.1. For all $\omega \in E^*$

$$m\Omega \leqslant \|\omega\|_{E^*} \qquad (m > 0) \tag{2.2}$$
$$\Omega = |w^*|, \quad \|f\|_{0,p,a}; \qquad f = w_x, w_y, u, v; \qquad a = G, \gamma$$

Here γ is a piecewise-smooth contour from $\bar{G}$, $1 \leqslant p < \infty$ and m is independent of the choice of ω but depends on $\{a, p\}$. Moreover, the ratio expressed by the inequality (2.2) possesses total continuity, i.e. from the boundedness of the set $\{\omega\}$ in E^* follows compactness in the sense of the left sides of (2.2).

Proof. The inequalities (2.2) are obtained by a unique method by using imbedding theorems, hence, we carry out the proof on one of them as an illustration

$$m|w^*| \leqslant \|w^*\|_{2,2} \leqslant \|\omega^*\|_E = \|\omega\|_{E^*}$$

Let us show the complete continuity of the ratio in this same example. Let the set $\{\omega: \|\omega\|_{E^*} \leqslant C\}$ be given. Consequently $\|\omega^*\|_E \leqslant C$, and taking into account the complete continuity of the imbedding $W_2^{(2)}(G^\circ) \to C(G^\circ)$, we obtain that $\{w^*\}$ is compact in $C(G^\circ)$. The assertion in the Lemma is proved.

Let us introduce a scalar product in the space E^*

$$(\omega_1, \omega_2)_H = A(\omega_1, \omega_2)$$

where the form A is defined by (2.1). Then

$$\|\omega\|_H^2 = 2B_1 \int_G \{(u_x + \nu v_y + \nu k w) u_x + (\nu u_x + v_y + kw)(v_y + kw) +$$
$$+ {}^1/_2 (1-\nu)(u_y + v_x)^2 + \alpha [w_x^2 + (w_y - kv)^2] + B_{21} [(w_{xx} + \nu w_{yy} - \nu k v_y) w_{xx} +$$
$$+ (\nu w_{xx} + w_{yy} - kv_y)(w_{yy} - kv_y) + 2(1-\nu)(w_{xy} - kv_x)^2]\} dG \tag{2.3}$$

Lemma 2.2. For all $\omega \in E^*$

$$m\|\omega\|_{E^*} \leqslant \|\omega\|_H \leqslant m_1 \|\omega\|_{E^*} \qquad (m, m_1 > 0) \tag{2.4}$$

Proof. According to (2.3), evidently

$$\|\omega'\|_H^2 \leqslant \text{const} \|\omega'\|_E^2$$

Passing to the exact lower bounds in all the elements $\omega' \in \omega$ we obtain that $\|\omega\|_H \leqslant m_1 \|\omega\|_{E^*}$. Let us prove the left side of the inequalities (2.4). It is actually necessary to show that

$$m = \inf (\|\omega\|_H \|\omega\|_{E^*}^{-1}) > 0, \qquad \omega \in E^*$$

Let us assume $m = 0$. In this case there exists a sequence $\{\omega_n\}$ such that $\|\omega_n\|_{E^*} = 1$, $\|\omega_n\|_H \to 0$, and $\omega_n \to \omega_0$ weakly in E^* as $n \to \infty$.

It turns out that the assumption that $\omega_0 = 0$ results in a contradiction. In fact, let $\omega_0 = 0$. From $\|\omega_n\|_H \to 0$ we obtain

$$\overset{\bullet}{u}_{n,x}, \ \overset{\bullet}{v}_{n,y} + k\overset{\bullet}{w}_n, \ \overset{\bullet}{u}_{n,y} + \overset{\bullet}{v}_{n,x}, \ \overset{\bullet}{w}_{n,xx}, \ \overset{\bullet}{w}_{n,yy} - k\overset{\bullet}{v}_{n,y}$$
$$\overset{\bullet}{w}_{n,xy} - k\overset{\bullet}{v}_{n,x} \to 0 \ \text{in} \ L_2(G^\circ) \tag{2.5}$$

and from the fact that $\omega_n \to 0$ weakly in E^*, and from Lemma 2.1 we have

$$|w_n^*|, \quad \|w_{n,x}^*\|_{0,2}, \quad \|w_{n,y}^*\|_{0,2} \to 0 \tag{2.6}$$

On the basis of (2.5), (2.6), we conclude

$$v_{n,y}^*, \ w_{n,xx}^*, \ w_{n,yy}^* \to 0 \text{ in } L_2(G^\circ)$$

The estimate of the mixed derivative [7] is known for functions from $W_2^{(2)}(G)$ and we present it in our terminology

$$\|w_{n,xy}^*\|_{0,2} \leqslant \text{const}\,(\|w_{n,xx}^*\|_{0,2} + \|w_{n,yy}^*\|_{0,2} + \|w_{n,x}^*\|_{0,2} + \|w_{n,y}^*\|_{0,2} + \|w_n^*\|_{0,2}) \tag{2.7}$$

As a result, $\|w_n^*\|_{2,2} \to 0$ from (2.6), (2.7). According to (2.4), it hence follows that $\|u_n^*\|_{1,2}, \ \|v_n^*\|_{1,2} \to 0$. This means that $\|\omega_n\|_{E^*} \to 0$ as $n \to \infty$ ($\|\omega_n\|_{E^*} = 1$). There is a contradiction, hence $\omega_0 \neq 0$.

We have $\omega_n = \omega_0 + \varphi_n$, where $\varphi_n = \omega_n - \omega_0$ and $\varphi_n \to 0$ weakly in E^*. By assumption

$$\|\omega_0 + \varphi_n\|_{H}^2 = \|\omega_0\|_{H}^2 + 2(\omega_0, \varphi_n)_H + \|\varphi_n\|_{H}^2 \tag{2.8}$$

Since

$$|(\omega_0, \varphi_n)_H| \leqslant \|\omega_0\|_H \|\varphi_n\|_H \leqslant \text{const}\,\|\omega_0\|_{E^*}\|\varphi_n\|_{E^*}$$

then $(\omega_0, \varphi)_H$ is a linear functional in E^* and the product $(\omega_0, \varphi_n)_H \to 0$ for $\varphi_n \to 0$ weakly in E^*. From this and from (2.8) we obtain $\|\omega_0\|_H^2 + \|\varphi_n\|_H^2 \to 0$ as $n \to \infty$. This time we arrive at a contradiction to $\omega_0 \neq 0$. The assertion of the Lemma is proved.

According to Lemma 2.2, the norms $\|\cdot\|_H$ and $\|\cdot\|_{E^*}$ are equivalent; the space E^* with the norm $\|\cdot\|_H$ will be called the space H. The following lemma is valid.

Lemma 2.3. The assertion of Lemma 2.1 remains valid if H replaces E^* throughout in the formulation.

3. Generalized solution. Let us assume the following conditions to be satisfied:

$$X, Y, Z_1 \in L_q(G), \qquad Z \in L(G), \qquad T^\circ, k_1 \in L_q(\gamma_1^+)$$

$$S^\circ, k_2 \in L_q(\gamma_2^+), \qquad N^\circ, k_3 \in L(\gamma_3^+), \qquad M^\circ, k_4 \in L_q(\gamma_4^+) \qquad (q > 1) \tag{3.1}$$

Here q is some fixed number. It is known that the equilibrium condition can be expressed by using the principle of virtual displacements

$$\Lambda(\omega, \omega^\circ) \equiv \int_G \{T_1(u_x^\circ + w_x w_x^\circ) + T_2[v_y^\circ + kw^\circ + (w_y - kv)(w_y^\circ - kv^\circ)] +$$
$$+ T_{12}[u_y^\circ + v_x^\circ + w_x(w_y^\circ - kv^\circ) + w_x^\circ(w_y - kv)] -$$
$$- M_1 w_{xx}^\circ - M_2(w_{yy}^\circ - kv_y^\circ) - 2M_{12}(w_{xy}^\circ - kv_x^\circ) +$$
$$+ (X - w_x Z_1)u^\circ + [Y - (w_y - kv)Z_1]v^\circ + (Z + Z_1)w^\circ\}\,dG -$$

$$- \int_{\gamma_1^+} (T^\circ - k_1 u) u^\circ d\gamma - \int_{\gamma_2^+} (S^\circ - k_2 v) v^\circ d\gamma - \int_{\gamma_3^+} (N^\circ - k_3 w) w^\circ d\gamma + \\ + \int_{\gamma_4^+} (M^\circ - k_4 w_x) w_x{}^\circ d\gamma = 0 \tag{3.2}$$

Here ω° is a virtual displacement, and the curvilinear integrals are written taking into account the customary rule of traversing the contour.

Definition 3.1. Let the function $\omega \in E$, which satisfies (3.2) for all $\omega^\circ \in E$, the generalized solution of the problem (1.6), (1.7) under the condition (3.1).

It is easy to see that each member of (3.2) is meaningful under these conditions. To do this it is sufficient to estimate each member in (3.2) by using the Hölder inequality and Lemma 2.3.

Let us show that (3.2) is equivalent to some operator equation in H. Indeed, according to (3.2), (1.1) we can write the following equation for the representatives $\omega^*, \omega^{\circ *}$ of the classes ω and ω° :

$$\Lambda(\omega^*, \omega^{\circ*}) = 2B_1 \int_G \{(u_x{}^* + \nu v_y{}^* + \nu k w^*) u_x{}^{\circ*} + (v_y{}^* + k w^* + \\ + \nu u_x{}^*)(v_y{}^{\circ*} + k w^{\circ*}) + {}^1/_2 (1-\nu)(u_y{}^* + v_x{}^*)(u_y{}^{\circ*} + v_x{}^{\circ*}) + \\ + B_{21} [(w_{xx}{}^* + \nu w_{yy}{}^* - \nu v_y{}^*) w_{xx}^{\circ}{}^* + (\nu w_{xx}{}^* + w_{yy}{}^* - k v_y{}^*) \times \\ \times (w_{yy}{}^{\circ*} - k v_y{}^{\circ*}) + 2(1-\nu)(w_{xy}{}^* - k v_x{}^*)(w_{xy}{}^{\circ*} - k v_x{}^{\circ*})] + \\ + \alpha [w_x{}^* w_x{}^{\circ*} + (w_y{}^* - k v^*)(w_y{}^{\circ*} - k v^{\circ*})]\} dG + \\ + \Big\langle 2B_1 \int_G \{-\alpha [w_x{}^* w_x{}^{\circ*} + (w_y{}^* - k v^*)(w_y{}^{\circ*} - k v^{\circ*})] + \\ + [u_x{}^* + {}^1/_2 w_x{}^{*2} + \nu v_y{}^* + \nu k w^* + {}^1/_2 \nu (w_y{}^* - k v^*)^2] w_x{}^* w_x{}^{\circ*} + \\ + {}^1/_2 [w_x{}^{*2} + \nu (w_y{}^* - k v^*)^2] u_x^{\circ}{}^* + [\nu u_x{}^* + {}^1/_2 \nu w_x{}^{*2} + v_y{}^* + k w^* + \\ + {}^1/_2 (w_y{}^* - k v^*)^2](w_y{}^* - k v^*)(w_y{}^{\circ*} - k v^{\circ*}) + {}^1/_2 [\nu w_x{}^{*2} + (w_y{}^* - k v^*)^2](v_y{}^{\circ*} + k w^{\circ*}) + \\ + {}^1/_2 (1-\nu)[u_y{}^* + v_x{}^* + w_x{}^*(w_y{}^* - k v^*)][w_x{}^*(w_y{}^{\circ*} - k v^{\circ*}) + \\ + w_x{}^{\circ*}(w_y{}^* - k v^*)] + {}^1/_2 (1-\nu)[w_x{}^*(w_y{}^* - k v^*)(u_y{}^{\circ*} + v_x{}^{\circ*})]\} dG \Big\rangle + \\ + \int_G \{(X - w_x{}^* Z_1) u^{\circ*} + [Y - (w_y{}^* - k v^*) Z_1] v^{\circ*} + (Z + Z_1) w^{\circ*}\} dG - \\ - \int_{\gamma_1^+} (T^\circ - k_1 u^*) u^{\circ*} d\gamma - \int_{\gamma_2^+} (S^\circ - k_2 v^*) v^{\circ*} d\gamma - \\ - \int_{\gamma_3^+} (N^\circ - k_3 w^*) w^{\circ*} d\gamma + \int_{\gamma_4^+} (M^\circ - k_4 w_x{}^*) w_x{}^{\circ*} d\gamma \doteq 0 \tag{3.3}$$

Henceforth, only normal representatives will appear throughout in the expressions, hence we omit the asterisk. Let us use the representation

$$\Lambda(\omega, \omega^\circ) = (\omega, \omega^\circ)_H + Q_1(\omega, \omega^\circ) + Q_2(\omega, \omega^\circ) \tag{3.4}$$

where $Q_1(\omega, \omega^\circ)$ is a member from (3.3) in the brackets $\langle \ldots \rangle$.

Lemma 3.1. The functional

$$P_1(\omega) = B_1 \int_G \{u_x w_x{}^2 + {}^1/_4 w_x{}^4 + v_y (w_y - kv)^2 + kw (w_y - kv)^2 +$$

$$+{}^{1}/_{4}\cdot(w_y - kv)^4 + \nu u_x(w_y - kv)^2 + \nu\,(v_y + kw)w_x^2 + {}^{1}/_{2}\nu w_x^{\,2}\,(w_y - kv)^2 + \\ + (1-\nu)\,(u_y + v_x)\,w_x\,(w_y - kv) + {}^{1}/_{2}\,(1-\nu)w_x^2\,(w_y - kv)^2\}\,dG \tag{3.5}$$

is weakly continuous in H.

Proof. Let $\omega_n \to \omega_0$ weakly in H, then by Lemma 2.3

$$\| w_{n,x}^2 - w_{0,x}^2 \|_{0,2},\quad \| (w_{n,y} - kv_n)^2 - (w_{0,y} - kv_0)^2 \|_{0,2} \to 0.$$

It is known that $(f_n, \varphi_n) \to (f_0, \varphi_0)$ if $f_n \to f_0$ in the weak, and $\varphi_n \to \varphi_0$ in the strong sense in Hilbert space. Applying this reasoning in (3.5) we obtain $P_1(\omega_n) \to P_1(\omega_0)$ for $\omega_n \to \omega_0$ weakly in H. Therefore, $P_1(\omega)$ is weakly continuous in H

It can be seen by direct verification that $Q_1(\omega, \omega^\circ)$ is a Gateau differential of $P_1(\omega)$. The following estimate holds

$$| Q_1(\omega, \omega^\circ) | \leqslant \text{const}\,(\| \omega \|_H^3 + \| \omega \|_H^2 + \| \omega \|_H)\,\| \omega^\circ \|_H$$

Hence $Q_1(\omega, \omega^\circ)$ is a linear functional in ω° in H, and by the Riesz lemma the following representation is valid

$$(\text{grad}_H P_1(\omega), \omega^\circ)_H = Q_1(\omega, \omega^\circ) \tag{3.6}$$

Note. The operator $\text{grad}_H\, P_1(\omega)$ is strongly continuous in H, as follows from the theorem of E.S. Tsitlanadze [8] on the complete continuity of potential operators.

Lemma 3.2. The representative

$$Q_2(\omega, \omega^\circ) = (K_2\omega, \omega^\circ) \tag{3.7}$$

is valid, where $K_2\omega$ is a strongly continuous operator in H.

Let us first prove that (3.7) holds. It is easy to obtain the inequality

$$|Q_2(\omega, \omega^\circ)| \leqslant \text{const}\,(\|\omega\|_H + \text{const})\|\omega^\circ\|_H$$

where the constants are positive, and independent of ω, ω°. It is hence seen that $Q_2(\omega, \omega^\circ)$ is a linear functional in ω° and the Riesz lemma verifies (3.7). Now let $\omega_n \to \omega_0$ weakly in H. Taking account of (3.1) and applying the Hölder inequality, we can write the following:

$$\begin{aligned} |(K_2\omega_n - K_2\omega_0, \omega^\circ)_H| &= | Q_2(\omega_n, \omega^\circ) - Q_2(\omega_0, \omega^\circ) | \leqslant \\ &\leqslant \text{const}\,\{\| w_{n,x} - w_{0,x} \|_{0,q_1} + \| w_{n,y} - w_{0,y} \|_{0,q_1} + \\ &+ \| v_n - v_0 \|_{0,q_2} + \| u_n - u_0 \|_{q_2,1} + \| v_n - v_0 \|_{q_2,2} + \\ &+ \| w_{n,x} - w_{0,x} \|_{q_3,4} + | w_n - w_0 |\}\,\| \omega^\circ \|_H \end{aligned} \tag{3.8}$$

where $q_1, q_2, q_3 > 1$ are some indices. Taking account of (3.8), the weak convergence to ω_n, Lemma 2.3, and the following norm definition

$$\|K_2\omega_n - K_2\omega_0\|_H = \sup|(K_2\omega_n - K_2\omega_0, \omega^\circ)_H|, \qquad \|\omega^\circ\|_H = 1$$

we obtain that $K_2\omega_n \to K_2\omega_0$ in H if $\omega_n \to \omega_0$ weakly. The Lemma is proved.

Let us substitute (3.6) and (3.7) into (3.4)

$$\Lambda(\omega, \omega^\circ) = (\omega - K\omega, \omega^\circ)_H, \qquad K = -\mathrm{grad}_H P_1(\omega) - K_2(\omega)$$

Therefore. (3.2) is equivalent to the equation

$$(\omega - K\omega, \omega^\circ)_H = 0 \qquad (\omega^\circ \in H) \tag{3.9}$$

or an operator equation with the completely continuous operator

$$\omega - K\omega = 0 \tag{3.10}$$

4. Existence of the generalized solution. The complete continuity of the operator K in H permits application of the Schauder-Leray method to prove the existence of a generalized solution of the problem. The scheme for utilization of this method in the theory of solvability of boundary value problems for shallow shells has been developed in [3], hence, we shall not dwell on terminology. Let $S(R,0)$ be a sphere of radius R with center at zero

$$S(R, 0) = \{\omega : \|\omega\|_H = 1\}$$

It turns out to be sufficient to prove that the completely continuous vector field $\omega - K\omega$ is homotopic on the sphere $S(R, 0)$ of sufficiently large radius R to a completely continuous vector field $I\omega$, where I is the identity operator, for which the rotation is $+1$ [8] . For homotopy it is sufficient to show that

$$(I - tK)\omega \neq 0 \qquad \text{for } \omega, t \in S(R, 0) \times [0, 1] \tag{4.1}$$

and R is arbitrary. Here t is a real parameter. When (4.1) is valid, it follows from the theory of the Schauder-Leray method [8] that a solution of (3 1) exists in the sphere $\|\omega\|_H < R$. In this section, it will essentially be proved that condition (4.1) is satisfied. Establishment of the inequality (4.1) is one of the basic propositions of this proof. The method proposed in [3] is utilized here to obtain it.

Let us introduce

$$e_1 = u_x, \qquad e_2 = v_y + kw, \qquad e_{12} = u_y + v_x \tag{4.2}$$

$$\theta_1 = {}^1/_2\, w_x^2, \qquad \theta_2 = {}^1/_2\,(w_y - kv)^2, \qquad \theta_{12} = w_x(w_y - kv)$$

$$\mathbf{a} = (a_1, a_2, a_{12}), \qquad \mathbf{a} = \mathbf{e}, \boldsymbol{\theta}, \boldsymbol{\varkappa}, \boldsymbol{\varepsilon}, \mathbf{T}, \mathbf{M}$$

$$\theta^+ = \theta R, \quad R > 0, \quad a_+ = R^{-1}a, \quad a = X, \ Y, \ldots, M^\circ$$

$$\Phi(\omega) = \Lambda(\omega, \omega) = (\omega - K, \omega, \omega)_H, \quad \|\omega\|_H = R$$

$$\Phi^+(R, \omega_+) = R^{-2}\Phi(\omega), \qquad \|\omega_+\|_H = 1$$

Taking account of (3.2), let us write the expression

$$\Phi(\omega) = 2\langle\mathbf{T}, \boldsymbol{\varepsilon}\rangle + \langle\mathbf{M}, \boldsymbol{\varkappa}\rangle - \int_G [T_1 u_x + T_{12} u_y + T_2 e_2 + T_{12} v_x]\, dG +$$

$$+\int_G \{(X - w_x Z_1) u + [Y - (w_y - kv) Z_1] v + (Z + Z_1) w\} dG -$$
$$-\int_{\gamma_1^+} (T^\circ - k_1 u) u \, d\gamma - \int_{\gamma_2^+} (S^\circ - k_2 v) v d\gamma - \int_{\gamma_3^+} (N^\circ - kw) w d\gamma +$$
$$+\int_{\gamma_4^+} (M^\circ - k_4 w_x) w_x d\gamma$$
$$\langle \mathbf{a}, \mathbf{b}\rangle = \int_G (a_1 b_1 + a_2 b_2 + 2a_{12} b_{12}) dG, \qquad \mathbf{a} = \mathbf{T}, \mathbf{M}, \quad \mathbf{b} = \boldsymbol{\varepsilon}, \boldsymbol{\varkappa} \tag{4.3}$$

If the quantity θ in the formulas for $\mathbf{T}, \boldsymbol{\varepsilon}$ in terms of $\mathbf{e}, \theta$ is provided with a plus superscript, then we mark $\mathbf{T}, \boldsymbol{\varepsilon}$ with the same superscript. Hence

$$\Phi^+(R, \omega) = 2\langle \mathbf{T}^+, \boldsymbol{\varepsilon}^+\rangle + \langle \mathbf{M}, \boldsymbol{\varkappa}\rangle - \int_G [T_1^+ e_1 + T_{12}^+ e_{12} + T_2^+ e_2] dG +$$
$$+\int_G \{(X_+ - w_x Z_1) u + [Y_+ - (w_y - kv) Z_1] v + (Z_+ + Z_{1+}) w\} dG -$$
$$-\int_{\gamma_1^+} (T_+^\circ - k_1 u) u d\gamma - \int_{\gamma_2^+} (S_+^\circ - k_2 v) v \, d\gamma -$$
$$-\int_{\gamma_3^+} (N_+^\circ - k_3 w) w \, d\gamma + \int_{\gamma_4^+} (M_+^\circ - k_4 w_x) w_x \, d\gamma \tag{4.4}$$

Let $\gamma_i \subset \gamma_i^+$ denote the set in which $k_i < 0$ $(i = 1,2,3)$ almost everywhere. According to Lemma 2.3, there are constants c_1^*, c_2^*, c_3^* such that

$$\|u\|^2_{0, q_1, \gamma_1} \leqslant c_1^* \|\omega\|_H^2, \qquad \|v\|^2_{0, q_1, \gamma_2} \leqslant c_2^* \|\omega\|_H^2$$
$$|\omega|^2 \leqslant c_3^* \|\omega\|_H^2, \qquad \frac{1}{q} + \frac{2}{q_1} = 1$$

Let us require the following condition to be satisfied

$$c_1 \|k_1\|_{0, q, \gamma_1} + c_2 \|k_2\|_{0, q, \gamma_2} + c_3 \|k_3\|_{0, 1, \gamma_3} < {}^1/_2 \tag{4.5}$$

where $c_i > c_i^*$ $(i = 1, 2, 3)$, and q correspond to (3.1).

Lemma 4.1. If conditions (3.2), (4.5) are satisfied, then there exists a positive constant σ independent of the choice of R and ω, such that for all sufficiently large R

$$\Phi(\omega) > \sigma R^2 \qquad (\omega \in S(R, 0)) \tag{4.6}$$

Proof. The following

$$\Phi^+(R, \omega) \geqslant \sigma \qquad (\omega \in S(1, 0)) \tag{4.7}$$

is equivalent to the latter assertion.

It is evidently sufficient to prove that

$$\lim \Phi^+ (R, \omega) > 0 \text{ for } R \to \infty \ (\omega \in S(1, 0)) \tag{4.8}$$

uniformly in ω, then there is a constant o and (4.7) holds for sufficiently large R. Indeed, otherwise there would be a sequence $\{R_m, \omega_n\}$ such that

$$\Phi^+ (R_m, \omega_n) \to c, \quad R_m \to \infty \quad \text{for } n,m \to \infty \quad c \leqslant 0, \quad \omega_n \in S(1, 0) \tag{4.9}$$

This contradicts (4.8), hence, the sufficiency of (4.8) to prove the Lemma should be acknowledged.

Let us return to the proof of the inequality (4.8). Let us assume (4.8) to be violated. This means there exists a sequence $\{R_m, \omega_n\}$ such that (4.9) holds. Let us extract the highest member from the polynomial $\Phi^+ (R, \omega)$

$$a_4^+ (R, \omega) = 8B_1 \int_G (\theta_1^+ + \theta_2^+)^2 \, dG$$

If the sequence $\theta_1^+{}_{,n,m}$ of $\theta_2^+{}_{,n,m}$ is not bounded in $L_2(G)$, then

$$\Phi^+ (R_m, \omega_n) \to +\infty$$

and (4.9) does not hold. Hence the sequences $\theta_1^+{}_{,n,m}$, $\theta_2^+{}_{,n,m}$, $\theta_{12}^+{}_{,n,m}$ are bounded in $L_2(G)$ since it has been assumed that (4.9) is satisfied. The boundedness of $\theta_{1,}^+{}_{n,m}$ as $R \to \infty$ means that

$$\| \theta_{1,n,m} \|_{0,2}, \ \| \theta_{2,n,m} \|_{0,2} \to 0, \qquad n, m \to \infty$$

Consequently, we can consider that $\omega_n \to \omega_0$ weakly in H $(\theta_1(\omega_0) = \theta_2(\omega_0) = 0)$.

The sequence $\theta_1^+{}_{,n,m}$ bounded in $L_2(G)$ can be considered weakly convergent, and moreover, because of (4.9) it can be considered that a finite limit exists

$$\lim \|\theta_1^+{}_{,n,m}\|_{0,2} = b, \qquad n,m \to \infty$$

Thus, a double limit of the sequence $\{\|\theta_{1,}^+{}_{n,m}\|_{0,2}\}$ exists as $n,m \to \infty$ and a limit in n also exists and equals zero for each fixed m. Applying the theorem on a duplicate limit, we obtain

$$b = \lim_m \lim_n \|\theta_1^+{}_{,n,m}\|_{0,2} = 0, \quad n,m \to \infty$$

Similarly, it is proved that

$$\lim \|\theta_2^+{}_{,n,m}\|_{0,2} = \lim \| \theta_{12}^+{}_{,n,m} \|_{0,2} = 0, \qquad n,m \to \infty$$

Let us utilize the deductions obtained. Evidently for any $\varepsilon > 0$ there is a positive integer n_0 such that

$$\int_G a_{n,m} b_{n,m} \, dG < 10^{-3} \varepsilon \quad \text{при } n, m > n_0(\varepsilon)$$

$$a = \theta_1^+, \ \theta_2^+, \ \theta_{12}^+, \qquad b = e_1, e_2, e_{12}$$

Let us expand the expression for $\Phi^+(R, \omega)$ in (4.9)

$$\Phi^+(R_m, \omega_n) = 2\|\omega_n\|_H^2 - \langle M_n, \varkappa_n\rangle - 2B_1 \int_G \{(e_{1,n} + \nu e_{2,n}) e_{1,n} + (\nu e_{1,n} + e_{2,n}) e_{2,n} +$$
$$+ 1/2 (1-\nu) e_{12,n}^2\} dG + \int_{\gamma_1^+} k_1 u^2 d\gamma + \int_{\gamma_2^+} k_2 v^2 d\gamma + \int_{\gamma_3^+} k_3 w^2 d\gamma + f(R_m, \omega_n)$$

Applying the Hölder inequality, and utilizing condition (4.5), we can hence deduce the following:

$$\Phi^+(R_m, \omega_n) \geqslant 1/2 \|\omega_n\|^2_H + f(R_m, \omega_n), \qquad n, m > n^0(\varepsilon)$$
$$|f(R_m, \omega_n)| < \varepsilon, \qquad n, m > n_0(\varepsilon), \qquad \|\omega_n\|_H = 1$$

The inequality obtained explicitly contradicts the assumption (4.9), and therefore, the validity of the assertion in the Lemma hence results logically.

L e m m a 4.2. If conditions (3.1), (4.5) are satisfied, then condition (4.1) is also satisfied.

P r o o f. Let us assume the Lemma to be false. In this case, for any R there are t, ω such that

$$\omega - tK\omega = 0, \qquad t \in [0,1], \qquad \omega \in S(R, 0) \tag{4.10}$$

Let us construct the functional

$$\Phi_1(t, \omega) = (\omega - tK\omega, \omega)_H, \qquad \Phi_1(0, \omega) = \|\omega\|^2_H, \qquad \Phi_1(1, \omega) = \Phi(\omega)$$

Taking into account the linearity of $\Phi_1(t, \omega)$ in t, we obtain

$$\Phi_1(t, \omega) = (1-t)\Phi_1(0, \omega) + t\Phi_1(1, \omega) = (1-t)\|\omega\|^2_H + t\Phi(\omega)$$

Because of Lemma 4.1 for sufficiently large R there should be

$$\Phi_1(t, \omega) \geqslant (1-t)\|\omega_H\|^2 + t\sigma\|\omega\|^2_H = (1 - t + t\sigma)\|\omega\|_H^2 \geqslant \min(1, \sigma)\ \omega_H^2 \tag{4.11}$$

On the other hand, (4.10) holds by assumption and

$$\Phi_1(t, \omega) = (\omega - tK\omega, \omega)_H = 0$$

Therefore, the assumption that the Lemma is false leads to (4.10), and this contradicts (4.11). The Lemma is proved.

L e m m a 4.3. Rotation of the completely continuous field $(I - K)\omega$ is $+1$ on a sphere $S(R, 0)$ of sufficiently large radius.
The assertion in this Lemma follows from Lemma 4.2 and reasoning expressed at the beginning of the Section.
In substance, Lemmas 4.1 and 4.2 prove the following theorem.

T h e o r e m 4.1. Let conditions (3.1) and (4.5) be satisfied. In this case a solution of the problem (1.6), (1.7) from H exists in a sphere of sufficiently large radius R.

Differential properties of the solution can be studied by utilizing the results obtained. Lemma 4.3 assures an approximate solution of (3.10) according to theorems in [8] on the convergence of the Galerkin method, and of other projection methods.

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INTEGRAL CRITERION OF STABILITY FOR SYSTEMS WITH QUASICYCLIC COORDINATES AND ENERGY RELATIONS FOR OSCILLATIONS OF CURRENT-CARRYING CONDUCTORS

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K. SH. KHODZHAEV
(Leningrad)
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We consider systems with quasicyclic coordinates and analyze the motions in which velocities, impulses and position (but not quasicyclic) coordinates are periodic functions of time. We assume that the generalized forces corresponding to quasicyclic coordinates either depend on time only, or are proportional to quasicyclic generalized coordinates and that the latter are small.

We show that, when certain requirements are imposed on the nonpotential forces with reference to the position coordinates in stable motions, then the quasicyclic impulses assume (up to the small order terms) mean values yielding the minimum of some function Λ of these mean values. This function can be expressed in terms of the Routh's kinetic potential of the system, by the virial describing the forces acting upon the position subsystem by the quasicyclic subsystem, etc. This in turn yields various versions of the integral criterion of stability.

Applying this criterion to the case of the oscillations of linear current-carrying conductors, we can relate mean periodic values of the magnetic fluxes to the extremal conditions of the combination of the averaged values of the magnetic field energy, magnetization energy and of the mechanical kinetic potential (or the virial of the ponderomotive forces).

The case when the Routh's equations are linear with respect to the position coordinates is considered separately, and we refer back to our previous papers on the problems